Cryptocurrencies are for Daring Investors

Ivilina Popova

Department of Finance & Economics, McCoy College of Business Administration, Texas State University, San Marcos, TX 78666

Abstract

We show that large allocation in cryptocurrencies can be explained by Prospect theory. Using piecewise linear value function, we show optimal portfolio construction with eleven asset classes including two cryptocurrencies, Bitcoin and Ethereum. We introduce a new Monte Carlo simulation approach that attaches tails to observed empirical distributions. A combination of stochastic optimization and the new simulation method shows that loss aversion and lottery type behavior are the main drivers behind large allocations in the cryptocurrencies.

Keywords: Cryptocurrencies, Fintech, Prospect theory, Stochastic optimization

1. Introduction

It is a well-known fact that standard expected utility theory cannot explain observed investment decision making under uncertainty. One example is the large amount of capital that went into crypto markets in 2017 and 2018. Pulse of Fintech 2019[1], reports that the global private investment in blockchain and cryptocurrencies (cryptos) totaled to \$9.9 billion for years 2017 and 2018. Bitcoin price hit a high of \$19,783 in December 2017 before dropping to \$6,200 in February 2018. Normative utility theories cannot obtain such allocations as optimal. However, Prospect theory is a descriptive theory and allows modeling of the observed behavior. Value functions replace utilities and decision weights replace probabilities. We optimize one such value function in an attempt to identify the group of investors who may see large allocations in cryptocurrencies as optimal.

The drastic increase in crypto prices have made them a potential new group of financial assets that may have a role to play among asset classes. However, as of now, whether cryptos are qualified as an asset class per traditional criteria is debatable, see Kritzman (1999). Despite it has satisfied some of the criteria for asset class, their total value as of now is still not large enough in terms of world investable wealth to represent an asset class. Nevertheless, the return on cryptos, Bitcoin in particular, has outperformed most traditional asset classes without any close competitor. However, cryptos have also displayed enormous volatility, which is also second to none in recent

Email address: ip12@txstate.edu (Ivilina Popova)

[1] https://home.kpmg/xx/en/home/campaigns/2019/07/pulse-of-fintech-h1-2019.html.

history. The huge volatility was caused by the wide swings and crashes in prices, which is referred to as tail risk. While the tails of a normal distribution have very small probability of occurrence, some assets' return distributions are leptokurtic, i.e., having fat-tails, causing potentially huge losses and gains when prices experience extreme volatility. Extreme volatility in the returns of cryptocurrencies seems to be pervasive.

Bitcoin is a digital currency introduced by Nakamoto (2008) as an electronic peer-to-peer payment system with decentralized public validation of transactions using blockchain technology[2]. Bitcoins may be purchased through a variety of platforms such as Coindesk and Gemini. While Bitcoin is a relatively new alternative currency (asset), there are ways to effectively place a Bitcoin short sale.[3] Additionally, Bitcoin futures are already traded and options on Bitcoin futures are coming in 2020[4].

There are several main differences between Bitcoin and fiat digital currencies (dollars, euro, yen, etc.). The most often mentioned one is decentralization. Nobody "owns" and "controls" the network. Volunteers around the world maintain the network. The protocol resolves the famous "double spending" problem with digital currencies. In the current banking system, this job function is performed by the banks. They have the power to verify transactions. Bitcoin protocol is completely different. In place of one large centralized system with information, the identity of every transaction is verified and stored by nodes in the open network. The second difference is the limited supply of Bitcoins. Fiat currencies have theoretically unlimited supply - central banks has the power to issue as many as they want. The maximum number of Bitcoins is 21 million and it is estimated that this number will be reached by year 2140.

Bitcoin can be seen as an alternative to online banking and companies like PayPal, Venmo and Zelle. Ethereum is based on a different concept. It has the goal to replace internet third parties by using a blockchain. It expands on the Bitcoin idea going beyond the banking system. Any providers that maintain large centralized databases of information could potentially be viewed as targets. Ethereum idea is to replace servers and clouds with "nodes" run by volunteers. It goes back to the original ideas of decentralized Internet and supporting a new type of applications (dapps). To imagine Ethereum is to think of a "world computer". Nobody owns it but to support its functionality is costly. The network uses "ether" a unique computer code that can be used to pay for using its computational resources. Similar to Bitcoin, Ether is a digital asset. In crypto jargon, Ether is like a "digital oil" and the Ethereum transaction fees represent how much "gas" it takes to complete a certain action. There is no limit to how many Ether tokens can be mined. As of today, there are no mechanisms to place a short sale on Ether. However, the CFTC chair said that Ether futures are likely in

[2] See, for example, `https://www.coindesk.com/information/`

[3] *Onecontracts* are offered by TF Global Markets in London. Another option to place a short sale is via margin trading. Bitfinex, for example, requires initial equity of 30% of the position to be able to short Bitcoin. One can also short a Bitcoin ETN (Exchange Traded Note) like *Bitcoin Tracker One*. Additionally, new structured products (Short Notes) are developed by the Swiss product houses Vontobel AG and Leonteq Securities AG.

[4] `https://www.cmegroup.com/trading/cryptocurrency-indices.html`.

2020[5].

Observing the interest of individual investors, venture capital firms and private equity in blockchain/crypto assets, it is clear that cryptos are here to stay. We may not see the irrational exuberance from the 2017 but the extreme volatility may continue.

Insert Figure 1 and Figure 2 Here

Figure 1 shows the volatile behavior of Bitcoin with the highest price observed at the end of 2017. Figure 2 compares the histograms of daily returns for Bitcoin and the exchange traded fund (ETF), SPY, for the period September 2015 – September 2019. Clearly, Bitcoin returns have heavier tails compared to the tails of SPY daily returns. In the empirical section we provide summary statistics on a variety of indexes tracking different types of assets as well as Bitcoin and Ethereum.

The main goal of this paper is to identify under what conditions large investment in cryptos is optimal. To achieve this we pursue several goals. First, we use a piecewise linear value function rooted in Prospect theory, and show a methodology for obtaining optimal solutions under uncertain asset returns. In order to model the tails of cryptos, we introduce a new Monte Carlo simulation algorithm that attaches tails to observed empirical distributions. Finally, we show by using the described optimization model and simulation algorithm under what conditions large allocations in cryptos are optimal.

The paper is organized as follows. Section 2 introduces the piecewise linear value function, the stochastic optimization model and a new algorithm for simulating returns by attaching tails to empirical distribution. Section 3 identifies the optimal tail simulation model for eleven assets including two cryptos, Bitcoin and Ethereum. Section 4 shows optimal allocations for portfolios of the eleven assets. Section 5 concludes.

2. Stochastic Optimization Model with Piecewise Value Function

Consider a portfolio allocation problem with N assets that have random returns $\tilde{R} = (R_1, \ldots, R_N) \geq 0$. The random return of allocation ω is $\tilde{R}\omega \equiv \sum_{i=1}^{N} R_i \omega_i$. We assume that $\tilde{R}$'s distribution is known and that we can generate independent and identically distributed (i.i.d.) observations of $\tilde{R}$. The asset allocation problem is then

$$\max_{\omega \in \Omega} Eu(\tilde{R}\omega), \tag{1}$$

where $\Omega = \{\omega : \sum_{i=1}^{N} \omega_i = 1, \omega_i \geq 0, i = 1, \ldots, N\}$ and $u(\cdot)$ is a value function satisfying the conditions from Prospect theory, see Kahneman and Tversky (1979).

Our ability to solve model (1) rests, in large part, on our ability to evaluate $Eu(\tilde{R}\omega)$. It is usually impossible to evaluate $Eu(\tilde{R}\omega)$ exactly, even for a fixed allocation vector $\omega \in \Omega$, unless the distribution of $\tilde{R}$ is particularly simple. When exact computation is

[5] https://www.coindesk.com/cftc-chair-says-well-likely-see-ether-futures-in-6-months.

not viable, it is natural to replace $Eu(\tilde{R}\omega)$ with a sampling-based estimator that lends itself to computation. With $\tilde{R}^j$, $j = 1, \ldots, n$, i.i.d. as $\tilde{R}$, we can approximate (1) by:

$$\max_{\omega \in \Omega} \left[\bar{u}_n(\omega) \equiv \frac{1}{n} \sum_{j=1}^{n} u(\tilde{R}^j \omega) \right]. \tag{2}$$

So, we resort to an approximation in which the expected value function is replaced with a sample mean estimate. For a detailed discussion about constructing simulation-based approximations for stochastic programming problems with application in portfolio optimization see King and Jensen (1992), Morton et al. (2003), Morton et al. (2006), Partani et al. (2006) and Popova et al. (2007).

Details on solving, and assessing the quality of the solution to (2) are described in Morton et al. (2006) and Popova et al. (2007).

Kahneman and Tversky (1992) propose a value function that is a two-part power function. Figure 3 shows the plot of this function for the proposed values of the estimated parameters shown in their paper. Such function satisfies the requirement for concavity for gains, convexity for losses and loss aversion. The way to view such function is that it is a typical decision maker's value function. Different investors may have different values for the parameters of the two-part power function.

Insert Figure 3

In this paper we consider a piece-wise linear value function. First, the piecewise linear function is computationally easier to handle than nonlinear functions (like the two-part power function) when attempting to solve a stochastic optimization model. Second, Cumulative Prospect theory, Kahneman and Tversky (1992), identifies four patterns of risk aversion:

1. Risk aversion for gains when the outcome probability is high.
2. Risk seeking for losses when the outcome probability is high.
3. Risk seeking for gains when the outcome probability is low.
4. Risk aversion for losses when the outcome probability is low.

By changing the slopes of the linear pieces of our value function and values of the two benchmarks, we will look for instances of the above fourfold pattern of risk aversion.

2.1. Piecewise Linear Value Function

Let $r_1 < r_2$ are given benchmarks and $s_2 > s_1 > s_0 \geq 0$ are slopes governing a three-piecewise linear value function. The value function, $u(r)$, satisfies the following conditions:

$$u'(r) = s_2, r \in (-\infty, r_2); u'(r) = s_1, r \in (r_2, r_1); u'(r) = s_0, r \in (r_1, \infty)$$

Figure 4 shows a plot of one such value function for different values of the slopes, $s_2 = 1, 0.5, 0.1$ and 0.025.

Insert Figure 4 here

The approximating model when optimizing a piecewise linear value function for the model defined in (1) can be constructed as follows. Let $R^i, i = 1, \ldots, n$ be IID observations of the random return vector. Additional decision variables $y_2^i, y_1^i, y_0^i, i = 1, \ldots n$ represent returns that are associated with each of the three pieces of the value function, under realization i. Note that since $s_2 > s_1 > s_0 \geq 0$ and the objective is maximization, $y_1^i > 0$ only if $y_2^i = r_2^i$ and $y_0^i > 0$ only if $y_1^i = r_1^i - r_2^i$. The approximating model is:

$$
\begin{aligned}
&\max_{y,\omega} \frac{1}{n} \sum_{i=1}^{n} s_2(y_2^i - r_2^i) + s_1 y_1^i + s_0 y_0^i \\
&\omega' e = 1 \\
&\omega' R^i = y_2^i + y_1^i + y_0^i, i = 1, \ldots, n \\
&\omega' \geq 0 \\
&-\infty \leq y_2^i \leq r_2^i, i = 1, \ldots, n \\
&0 \leq y_1^i \leq r_1^i - r_2^i, i = 1, \ldots, n \\
&0 \leq y_0^i, i = 1, \ldots, n
\end{aligned}
\tag{3}
$$

The above formulation allows the return realizations, $\omega' R$, to take any real value (note that y_2^i is unrestricted in sign with its allowable values only being bounded above by r_2^i). The intercept of the value function is selected so that it is positive when the lower target is exceeded and negative when it is not. We will experiment with different values of the slope parameters to analyze the change in the optimal portfolio allocation, especially when we introduce fat tails through the simulation procedure discussed later in the paper.

2.2. Attaching Exponential Tails to Empirical Distribution

In order to be able to generate instances of the approximating model described in (3) we need to generate random returns from a multivariate distribution. The algorithm described in Morton et al. (2006) and Popova et al. (2007) uses Nelson and Cario (1997) approach for simulating returns from different distributions. In one of its steps we need a formulation of the marginal distributions. We do so by using a mixture of empirical/exponential distributions. The procedure described next is a modification of the method described in Section 5.2.4 in Bratley et al. (1987).

We show two modifications, the first one attaches exponential tails on the left and right sides of the empirical distribution. The second one attaches only a left tail. Suppose that we have n observations (returns). First, we order them in increasing order so that $X_1 < X_2 < \ldots < X_n$. For the two tail modification we fit a piecewise linear CDF for the values between X_k and X_{n-k}, where k can be chosen to be the 5^{th} or 1^{st} percentile of the empirical return distribution. Then the two tail modified CDF is:

$$F(t) = \begin{cases} k/n \times exp[(X_k - t)/\Theta_L], \text{for } t < X_k \\ \\ i/n + (t - X_i)/[n(X_{i+1} - X_i)], \\ \qquad\qquad \text{for } X_i \le t \le X_{i+1}, i = k+1, \cdots n-k-1 \\ \\ 1 - k/n \times exp[-(t - X_{n-k})/\Theta_R], \text{for } t > X_{n-k} \end{cases} \tag{4}$$

Where

$$\Theta_R = \left[\frac{X_{n-k}}{2} + \sum_{i=n-k+1}^{n} (X_i - X_{n-k}) \right] /k \tag{5}$$

and

$$\Theta_L = \left[\frac{X_k}{2} + \sum_{i=1}^{k-1} (X_i - X_k) \right] /k. \tag{6}$$

To generate a variate from this two tailed mixed distribution by inversion can be done by following the next steps:

1. Generate a random U, Uniform on $(0, 1)$;
2. If $U > 1 - k/n$ then return $X = X_{n-k} - \Theta_R \times ln[n(1 - U)/k]$;
3. If $U < k/n$, then return $X = X_k - \Theta_L \times ln[nU/k]$;
4. Otherwise set $V \leftarrow nU, I \leftarrow \lfloor V \rfloor$, and return $X = (V - I)(X_{I+1} - X_I) + X_I$.

For the left tail modification, we fit a piecewise linear CDF for the values between X_{k+1} and X_n, where k can be chosen to be the 5^{th} or 1^{st} percentile of the empirical return distribution. Then the left tail modified CDF is:

$$F(t) = \begin{cases} k/n \times exp[(X_k - t)/\Theta_L], \text{for } t < X_k \\ \\ i/n + (t - X_i)/[n(X_{i+1} - X_i)], \\ \qquad\qquad \text{for } X_i \le t \le X_{i+1}, i = k+1, \cdots n \end{cases} \tag{7}$$

Where

$$\Theta_L = \left[\frac{X_k}{2} + \sum_{i=1}^{k-1} (X_i - X_k) \right] /k. \tag{8}$$

To generate a variate from this left tail mixed distribution by inversion can be done by following the next steps:

1. Generate a random U, Uniform on $(0, 1)$;
2. If $U < k/n$, then return $X = X_k - \Theta_L \times ln[nU/k]$;
3. Otherwise set $V \leftarrow nU, I \leftarrow \lfloor V \rfloor$, and return $X = (V - I)(X_{I+1} - X_I) + X_I$.

3. Modeling Tails for Daily Returns

For the empirical experiment we select the following Exchange Traded Funds:

- GLD: SPDR Gold Shares,
- HYG: iShares iBoxx High Yield Corporate Bond,
- BND: Vanguard Total Bond Market,
- GSG: iShares S&P GSCI Commodity-Indexed Trust,
- VNQ: VANGUARD Real Estate Index Fund,
- UUP: Invesco DB US Dollar Index Bullish Fund,
- SPY: SPDR S&P 500 trust,
- IWM: iShares Russell 2000,
- EFA: iShares MSCI EAFE Index Fund,
- BTH: Bitcoin,
- ETH: Ethereum.

The goal is to have representations from a variety of asset classes. The above selection includes, gold, high yield corporate bonds, total bond market, commodities, real estate, US Dollar, US stock market, small cap stocks, international stocks and cryptos. Daily prices for all ETFs are downloaded from Bloomberg while daily prices for Bitcoin and Ethereum are obtained from Yahoo!Finance. The selected time period is September 2015 – September 2019. Table 1 shows summary statistics for all assets.

Insert Table 1 Here

Note that EFA is the asset with the highest kurtosis, followed by Ethereum, SPY and Bitcoin. Additionally, EFA shows the largest negative skewness and eight out of eleven asset classes show negative skewness.

Previous studies (e.g., Elendner et al. (2018); Osterrieder et al. (2017); Wu and Pandey (2014)) have shown that returns of cryptos exhibit a high degree of asymmetry and occurrence of extreme events. Wu and Pandey (2014) find that Bitcoin's return distribution was leptokurtic and fat-tailed. Osterrieder et al. (2017) find that cryptocurrencies exhibit strong non-normal characteristics, large tail dependencies, depending on the particular cryptocurrencies and heavy tails. They investigated the tail dependence of cryptocurrencies using both empirical and Gaussian copulas.

To investigate what is the best simulation model to use, we apply the procedure described in the previous section that allows for introducing tails in empirical distributions to the daily historical returns for all eleven ETFs. To understand the impact of the choice of the cutoff points used in (4) and (7), we simulate 10,000 scenarios by using the 5^{th}, 1^{st} and 0.1^{st} percentiles as cutoff points from assets' empirical return distributions. Table 2 shows the choice of either left tailed or two tailed distribution and the corresponding cutoff points that bring the simulated and historical distributions as close as possible. The match uses the first four moments, the median, min and max values and minimizes the sum of squared differences. Table 3 shows summary statistics for the historical and simulated assets' returns for the best fit found[6] and shown in Table 2.

Insert Table 2 and Table 3 here

Results illustrate that the left tail modification with the cutoff point at the 5^{th} percentile works for majority of the assets. Exceptions are UUP, EFA, Bitcoin and Ethereum. The best fit for UUP comes from the two tails model with a cutoff at the 1^{st} percentile, EFA best fit is the two tail model with a cutoff at the 0.1^{st} percentile, Bitcoin and Ethereum best fit is the left tail model with a cutoff at the 1^{st} percentile.

4. Computing Optimal Portfolios

In this section we construct optimal portfolios using the approximating model described in (3) for the piecewise linear value function. To obtain optimal solutions we follow the combination of simulation and optimization steps from Morton et al. (2006) and Popova et al. (2007). The simulated returns are based on the new methods shown in (4) and (7). The optimization follows the procedure from Morton et al. (2006), page 511. They construct a six steps algorithm for identifying a solution for the portfolio allocation problem and establishing its quality. In our case we simply follow their steps by replacing the objective function with the piecewise linear value function.

There are several parameters that could influence the optimal allocation. The two benchmarks, r_1 and r_2, and the slopes, s_0, s_1 and s_2. If we fix $u(r_1) = 1$, and $s_0 = 0$, we get a probability like value function when $r > r_1$. If we set $u(r_2) = 0$, the value of s_1 can be computed as $s_1 = 1/(r_1 - r_2)$. The value of s_2 depends on how much we want to penalize the downside. Figure 4 shown earlier, has $u(r_1) = 1$, $s_0 = 0$, $u(r_2) = 0$ and $s_1 = 1/(r_1 - r_2)$. Additionally, the three linear equations describing the value function become:

1. $y = 1$ when $r > r_1$.
2. $y = r/(r_1 - r_2) - r_2/(r_1 - r_2)$ when $r_2 < r < r_1$.
3. $y = s_2 r - s_2 r_2$ when $r < r_2$.

Following the algorithm from Morton et al. (2006), we solve 30 instances of the approximating model (3) and compute the optimal weights as the averages of the 30 optimal solutions[7]. The optimal values of the objective function are used to compute an upper bound of the value function. Given the optimal weights, we generate 100,000

[6]Detailed results for all cutoff points and distributions are available upon request.

[7]The approximating models are generated using C++ code and solved with the academic version of IBM CPLEX solver.

scenarios and obtain a lower bound of the optimal value function. We report the lower bounds of the solutions.

Table 4 shows optimal results for a combination of parameter values. To obtain the optimal upper bound of the solution, the approximating models use scenarios with 1,000 returns by simulating from the mix of empirical and left tail distribution described in (7) with a cutoff point at the 0.1^{st} percentile. Four different values are used for s_2: 1, 0.5, 0.1 and 0.025. The value of the second benchmark r_2 is fixed at zero and the value of the first benchmark, r_1 varies from 1% to 9%.

Insert Table 4 here

Note first that as the value of s_2 increases, the loss aversion increases. This is visually represented in Figure 4 as well. In Table 4 we fix the value of the second benchmark, r_2, in order to analyze the behavior of the solution associated with "gains". We vary the value of the first benchmark, r_1 from 1% to 9%. Keep in mind that we are working with daily returns and we construct daily portfolios. Clearly a daily return of 9% is an extreme event. Daily return of 1% is still high but more within a normal range. Lets compare the optimal portfolios for fixed value of the slope s_2. For $s_2 = 1$ and $r_1 = 1\%$, the highest allocation is in the fixed income fund, BND. Given that the benchmark is not aggressive, and within highly probably range, this is evidence of risk aversion for gains when the outcome probability is high. However, when $r_1 = 9\%$, a very low probability event, the highest allocation switches to Bitcoin and Ethereum. Investor preferences switch from risk aversion to risk seeking in order to chase the rare event. This behavior was described as risk seeking for gains when the outcome probability is low and used to explain why people buy lottery tickets in Kahneman and Tversky (1992). We observe similar pattern for $s_2 = 0.5$ and $s_2 = 0.1$, i.e. initial high allocation in BND for $r_1 = 1\%$ and high allocation in cryptos for $r_1 = 9\%$. However, when $s_2 = 0.025$, there is no notable change in the optimal portfolio allocation for different values of the benchmark r_1. In our model, the triggers of the risk aversion/risk seeking preferences could come from the benchmarks and from the slope parameter, s_2. When s_2 is very small, there is almost no loss aversion and the investor does not exhibit a switch between risk aversion and risk seeking preferences. Table 4 showed optimal portfolio weights from a perspective of what happens in the "positive domain". We see that the highest allocations in cryptos occur when investors are after lottery gains.

Insert Table 5 here

Next, we move into analyzing the optimal weights in the "negative domain", i.e. what happens when investors are facing losses associated with missing the second benchmark, r_2. Table 5 shows optimal portfolio statistics and weights for different values of the two benchmarks r_1 and r_2, and the slope parameter, s_2. The risk seeking behavior and loss aversion associated with higher values for the slope parameter s_2 is clearly present. Allocations in cryptos increase as s_2 increases. For $s_2 = 0.025$, $r_2 = 0\%$, the optimal weight on Bitcoin is 0.39%. However, when $s_2 = 1$, the weight increases to 37.26%. Similar behavior is observed for the other values of the second benchmark, r_2. We can also observe the switch from risk seeking in the negative domain to risk aversion, the fourth pattern of risk attitudes in Prospect theory. Note that the highest weight for $s_2 = 0.1$ and $r_2 = 0\%$ is in the fixed income fund, BND. However, when $r_2 = 4\%$, the highest allocation is in Bitcoin, 30.19%. Clearly, as the value of the second benchmark becomes less extreme and as a result the probability of loss declines, risk preferences switch from risk seeking to risk averse. This is analogous to purchasing insurance discussed in Kahneman and Tversky (1992). In summary, in the "negative domain", highest allocation in cryptos are for risk seeking investors[8].

5. Conclusion

In this paper we show under what preferences large investment in cryptocurrencies are optimal. We use a Prospect theory value function, and an approximating model to solve the general investment stochastic optimization model. Our results show that high allocations in cryptos are optimal when investors become risk seekers or bet on lottery type event.

Future work currently in progress includes attaching a quadratic function in the "negative" domain in order to better capture the nonlinear nature of the loss aversion behavior of investors; a detailed sensitivity analysis identifying the effect of the optimal portfolio allocation when simulating based on different cutoff points; and a trading strategy using a sliding window of daily historical returns to compare the behavior of the portfolios representing the fourfold pattern of risk aversion.

[8] We have results for a wide range of benchmarks and slope values showing similar patterns of behavior. They are available upon request.

References

Bratley, P., Fox, B. L., and Schrage, L. E., 1987. A Guide to Simulation, Springer–Verlag.

Cario, M. C., and Nelson, B. L., 1997. Modeling and Generating Random Vectors with Arbitrary Marginal Distributions and Correlation Matrix, Technical Report, Department of Industrial Engineering and Management Sciences, Northwestern University.

Elendner, H., S. Trimborn, B. Ong, and T.M. Lee, 2018. The Cross-Section of Crypto-Currencies as Financial Assets: Investing in Crypto-Currencies Beyond Bitcoin. Handbook of Blockchain, Digital Finance, and Inclusion, Volume 1, pp. 145-173.

Kahneman, D., and Tversky, A., 1979. Prospect Theory: An Analysis of Decision under Risk. Econometrica, V.47, No. 2 (March), pp. 263-292.

Kahneman, D., and Tversky, A., 1992. Advances in Prospect Theory: Cumulative representation of Uncertainty. Journal of Risk and Uncertainty, V.5, pp. 297-323.

King, A. J., and Jensen, D. L., 1992. Linear-Quadratic Efficient Frontiers for Portfolio Optimization, Applied Stochastic Models and Data Analysis, Vol. 8, 195–207.

Kritzman, M., 1999. Toward Defining an Asset Class. Journal of Alternative Investments, 2(1), pp. 79-82.

Mak, W. K., Morton D. P., and Wood, R.K., 1999. Monte Carlo Bounding Techniques for Determining Solution Quality in Stochastic Programs, Operations Research Letters, Vol. 24, 47–56.

Morton, D. P., Popova, E., and Popova, I., 2006. Efficient Fund of Hedge Funds Construction under Downside Risk Measures, Journal of Banking and Finance, Vol. 30, 503–518.

Morton, D. P., Popova, E., Popova, I., and Zhong, M., 2003. Optimizing Benchmark-based Utility Functions. Bulletin of the Czech Econometric Society, Vol. 10, 1–18.

S. Nakamoto, 2008. Bitcoin: A peer-to-peer electronic cash system, `https://bitcoin.org/bitcoin.pdf`.

Osterrieder, J., Lorenz, J., and Strika, M., 2017. Bitcoin and Cryptocurrencies – Not for Faint-Hearted. International Banking and Finance, V.4, No.1, pp. 56-94.

Partani, A., Morton, D. P., and Popova, I., 2006. Jackknife Estimators for Reducing Bias in Asset Allocation, in Proceedings of the 2006 Winter Simulation Conference.

Popova, I., Morton, D. P., Popova, E., and Yau, J., 2007. Optimizing Benchmark-Based Portfolios with Hedge Funds, Journal of Alternative Investments, Vol. 10, 35-55.

Wu, C. Y., and Pandey, V.K., 2014. The Value of Bitcoin in Enhancing the Efficiency of an Investor's Portfolio. Journal of Financial Planning, 27(9), pp. 44-52.

Table 1: Daily Summary Statistics for Eleven Assets

	GLD	**HYG**	**BND**	**GSG**	**VNQ**	
Arithmetic Mean	0.0255%	0.0013%	0.0039%	-0.0147%	0.0231%	
Standard Deviation	0.7709%	0.3902%	0.1939%	1.1465%	0.9102%	
Skewness	0.3125	0.0010	-0.2809	-0.1710	-0.5109	
Kurtosis	2.9356	3.3568	1.2582	1.0313	1.5861	
Min	-3.5328%	-2.0167%	-0.9938%	-4.1409%	-4.0408%	
Max	4.7953%	1.6688%	0.6927%	4.4543%	3.2907%	
	UUP	**SPY**	**IWM**	**EFA**	**Bitcoin**	**Ethereum**
Arithmetic Mean	0.0064%	0.0403%	0.0262%	0.0088%	0.3638%	0.4777%
Standard Deviation	0.4120%	0.8368%	1.0361%	0.8831%	4.5904%	7.8240%
Skewness	-0.1470	-0.5480	-0.3490	-1.3384	-0.0856	0.5911
Kurtosis	3.0160	4.1063	1.6922	11.5348	4.0171	4.6298
Min	-2.4116%	-4.2722%	-4.3508%	-8.9772%	-24.5886%	-28.5872%
Max	2.5752%	4.9290%	4.6942%	3.0162%	22.7618%	49.7580%

The Exchange Traded Funds are: GLD: SPDR Gold Shares, HYG: iShares iBoxx High Yield Corporate Bond, BND: Vanguard Total Bond Market, GSG: iShares S&P GSCI Commodity Indexed Trust, VNQ: VANGUARD Real Estate Index Fund, UUP: Invesco DB US Dollar Index Bullish Fund, SPY: SPDR S&P 500 trust, IWM: iShares Russell 2000, EFA: iShares MSCI EAFE Index Fund, BTH: Bitcoin and ETH: Ethereum. Daily prices for all ETFs are downloaded from Bloomberg while daily prices for Bitcoin and Ethereum are obtained from Yahoo!Finance. The selected time period is September 2015 – September 2019.

Table 2: Best Fit Simulation Model

	Distribution	**Cut off**
GLD	Left Tail	5.0%
HYG	Left Tail	5.0%
BND	Left Tail	5.0%
GSG	Left Tail	5.0%
VNQ	Left Tail	5.0%
UUP	Two Tails	1.0%
SPY	Left Tail	5.0%
IWM	Left Tail	5.0%
EFA	Two Tails	0.1%
Bitcoin	Left Tail	1.0%
Ethereum	Left Tail	1.0%

We simulate 10,000 scenarios by using the 5^{th}, 1^{st} and 0.1^{st} percentiles as cutoff points from assets' empirical return distributions. Results show the choice of either left tailed or two tailed distribution and the corresponding cutoff points that bring the simulated and historical distributions as close as possible. The match minimizes the sum of squared differences based on the first four moments, the median and the min and max values.

Table 3: Historical and Simulated Moments

	GLD *Simulated*	**GLD** *Historical*	**HYG** *Simulated*	**HYG** *Historical*	**BND** *Simulated*	**BND** *Historical*
Min	-3.8670%	-3.5328%	-2.4305%	-2.0167%	-1.0947%	-0.9938%
Max	4.7463%	4.7953%	1.6687%	1.6688%	0.6892%	0.6927%
Average	0.0333%	0.0255%	0.0051%	0.0013%	0.0062%	0.0039%
StDev	0.7680%	0.7709%	0.3912%	0.3902%	0.1946%	0.1939%
Medan	0.0232%	0.0000%	0.0139%	0.0114%	0.0120%	0.0000%
Skewness	0.2006	0.3125	-0.0635	0.0010	-0.3415	-0.2809
Kurtosis	2.2654	2.9356	3.3004	3.3568	1.2467	1.2582
	GSG *Simulated*	**GSG** *Historical*	**VNQ** *Simulated*	**VNQ** *Historical*	**UUP** *Simulated*	**UUP** *Historical*
Min	-6.4938%	-4.1409%	-5.8063%	-4.0408%	-2.9854%	-2.4116%
Max	4.4188%	4.4543%	3.2617%	3.2907%	2.7958%	2.5752%
Average	-0.0017%	-0.0147%	0.0333%	0.0231%	0.0113%	0.0064%
StDev	1.1527%	1.1465%	0.9155%	0.9102%	0.4164%	0.4120%
Medan	0.0580%	0.0000%	0.0490%	0.0356%	0.0000%	0.0000%
Skewness	-0.2885	-0.1710	-0.6264	-0.5109	-0.2454	-0.1470
Kurtosis	1.3572	1.0313	2.0098	1.5861	2.8425	3.0160
	SPY *Simulated*	**SPY** *Historical*	**IWM** *Simulated*	**IWM** *Historical*	**EFA** *Simulated*	**EFA** *Historical*
Min	-6.5402%	-4.2722%	-6.6226%	-4.3508%	-9.2507%	-8.9772%
Max	4.8360%	4.9290%	4.6355%	4.6942%	7.3376%	3.0162%
Average	0.0471%	0.0403%	0.0370%	0.0262%	0.0192%	0.0088%
StDev	0.8438%	0.8368%	1.0420%	1.0361%	0.8897%	0.8831%
Medan	0.0482%	0.0380%	0.0706%	0.0364%	0.0326%	0.0158%
Skewness	-0.8610	-0.5480	-0.5107	-0.3490	-0.9455	-1.3384
Kurtosis	5.0747	4.1063	2.1759	1.6922	8.3610	11.5348
	Bitcoin *Simulated*	**Bitcoin** *Historical*	**Ethereum** *Simulated*	**Ethereum** *Historical*		
Min	-28.7043%	-24.5886%	-48.4620%	-28.5872%		
Max	22.6342%	22.7618%	49.1847%	49.7580%		
Average	0.3999%	0.3638%	0.5370%	0.4777%		
StDev	4.6271%	4.5904%	7.8700%	7.8240%		
Medan	0.4020%	0.3643%	0.0729%	0.0000%		
Skewness	-0.2166	-0.0856	0.3641	0.5911		
Kurtosis	4.0303	4.0171	4.4678	4.6298		

Table shows the historical and simulated daily asset moments based on the best match model shown in Table 2. The match minimizes the sum of squared differences based on the first four moments, the median and the min and max values. GLD, HYG, BND, GSG, NQ, SPY, IWM simulated returns use a mixture of their empirical distributions and exponential left tail based on a cutoff at the 5^{th} percentile; UUP simulated returns use a mixture of the empirical distribution and exponential two tails with cutoff at the 1^{st} percentile; EFA simulated returns use a mixture of the empirical distribution and exponential two tails with cutoff at the 0.1^{st} percentile; Bitcoin and Ethereum simulated returns use a mixture of the empirical distributions with an exponential left tail with cutoff at the 1^{st} percentile.

Table 4: Optimal portfolio statistics and weights for different values of benchmark r_1 and slope parameter s_2.

s_2	**Portfolio Mean**	**Portfolio STD**	r_1	r_2	**GLD**	**HYG**	**BND**	**GSG**	**VNQ**	**UUP**	**SPY**	**IWM**	**EFA**	**Bitcoin**	**Ethereum**
1	5.40%	53.79%	1%	0%	15.70%	2.28%	13.79%	1.33%	11.75%	21.30%	15.83%	7.58%	1.03%	6.88%	2.53%
1	13.81%	148.67%	3%	0%	18.89%	0.00%	4.11%	1.04%	12.62%	4.29%	14.29%	13.78%	0.48%	21.96%	8.54%
1	21.80%	247.47%	5%	0%	13.20%	0.00%	2.36%	1.49%	9.09%	1.99%	8.90%	10.88%	0.01%	37.51%	14.57%
1	29.73%	347.00%	7%	0%	8.29%	0.00%	0.40%	0.80%	4.55%	0.65%	5.17%	6.77%	0.00%	52.74%	20.62%
1	36.23%	429.91%	9%	0%	3.15%	0.00%	0.00%	0.10%	1.63%	0.01%	1.32%	3.10%	0.00%	64.43%	26.26%
0.5	1.32%	16.58%	1%	0%	4.50%	8.03%	57.54%	0.57%	1.93%	19.21%	3.83%	1.49%	1.37%	1.07%	0.46%
0.5	13.81%	148.67%	3%	0%	18.89%	0.00%	4.11%	1.04%	12.62%	4.29%	14.29%	13.78%	0.48%	21.96%	8.54%
0.5	21.80%	247.47%	5%	0%	13.20%	0.00%	2.36%	1.49%	9.09%	1.99%	8.90%	10.88%	0.01%	37.51%	14.57%
0.5	29.73%	347.00%	7%	0%	8.29%	0.00%	0.40%	0.80%	4.55%	0.65%	5.17%	6.77%	0.00%	52.74%	20.62%
0.5	36.23%	429.91%	9%	0%	3.15%	0.00%	0.00%	0.10%	1.63%	0.01%	1.32%	3.10%	0.00%	64.43%	26.26%
0.1	0.78%	14.32%	1%	0%	3.35%	10.34%	58.95%	0.76%	0.77%	17.02%	2.24%	1.10%	1.64%	0.36%	0.15%
0.1	0.95%	15.14%	3%	0%	3.73%	9.94%	60.27%	0.63%	1.12%	18.05%	2.77%	1.20%	1.49%	0.57%	0.24%
0.1	1.33%	16.60%	5%	0%	4.51%	8.05%	57.48%	0.57%	1.94%	19.22%	3.83%	1.49%	1.37%	1.08%	0.46%
0.1	5.65%	57.21%	7%	0%	9.06%	3.89%	32.34%	0.56%	7.32%	19.84%	10.71%	3.61%	1.46%	7.82%	3.38%
0.1	29.51%	344.05%	9%	0%	9.92%	0.00%	0.06%	0.97%	4.79%	0.27%	4.15%	6.83%	0.00%	53.21%	19.80%
0.025	0.77%	14.74%	1%	0%	3.44%	10.79%	61.37%	0.85%	0.68%	17.44%	2.16%	1.09%	1.73%	0.33%	0.13%
0.025	0.80%	14.79%	3%	0%	3.50%	10.69%	61.18%	0.82%	0.76%	17.49%	2.28%	1.10%	1.66%	0.36%	0.15%
0.025	0.82%	14.84%	5%	0%	3.52%	10.63%	61.05%	0.77%	0.79%	17.58%	2.37%	1.13%	1.62%	0.39%	0.16%
0.025	0.85%	14.89%	7%	0%	3.53%	10.50%	61.02%	0.72%	0.83%	17.57%	2.46%	1.16%	1.59%	0.43%	0.18%
0.025	0.89%	14.98%	9%	0%	3.60%	10.28%	60.73%	0.70%	0.94%	17.78%	2.57%	1.16%	1.56%	0.49%	0.20%

Table shows optimal results for four different values for s_2: 1, 0.5, 0.1 and 0.025. The value of the second benchmark r_2 is fixed at zero and the value of the first benchmark, r_1 varies from 1% to 9%. To obtain the optimal upper bound of the solution, the approximating models use scenarios with 1,000 returns by simulating from the mix of empirical and left tail distribution described in (7) with a cutoff point at the 0.1^{st} percentile. Table shows the optimal portfolios average daily returns, standard deviations and the optimal weights on the eleven assets.

Table 5: Optimal portfolio statistics and weights for different values of the two benchmarks r_1 and r_2, and the slope parameter, s_2.

s_2	**Portfolio Mean**	**Portfolio STD**	r_1	r_2	**GLD**	**HYG**	**BND**	**GSG**	**VNQ**	**UUP**	**SPY**	**IWM**	**EFA**	**Bitcoin**	**Ethereum**
1	21.82%	247.48%	5%	-4%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
1	21.82%	247.48%	5%	-2%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
1	21.82%	247.48%	5%	0%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
1	21.82%	247.48%	5%	2%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
1	21.82%	247.48%	5%	4%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
0.5	21.17%	239.93%	5%	-4%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	8.88%	0.00%	36.55%	14.00%
0.5	21.82%	247.48%	5%	-2%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
0.5	21.82%	247.48%	5%	0%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
0.5	21.82%	247.48%	5%	2%	14.26%	0.00%	1.89%	1.02%	9.20%	1.64%	9.21%	10.77%	0.00%	37.26%	14.73%
0.5	19.84%	222.55%	5%	4%	16.03%	0.00%	2.63%	1.07%	10.26%	1.29%	10.73%	11.40%	0.00%	33.34%	13.24%
0.1	20.09%	226.80%	5%	-4%	15.35%	0.00%	2.58%	1.13%	7.74%	1.37%	9.56%	11.09%	0.32%	33.99%	13.55%
0.1	14.95%	163.08%	5%	-2%	18.38%	0.00%	3.40%	1.38%	11.83%	5.40%	15.29%	11.08%	0.11%	22.48%	10.65%
0.1	1.35%	16.70%	5%	0%	4.68%	8.03%	57.09%	0.58%	2.04%	19.36%	3.95%	1.54%	1.18%	1.09%	0.47%
0.1	11.53%	120.98%	5%	2%	19.80%	0.00%	4.41%	1.27%	14.09%	6.06%	18.56%	11.14%	0.43%	17.35%	6.91%
0.1	18.16%	201.82%	5%	4%	16.85%	0.00%	2.66%	1.24%	10.65%	2.37%	11.46%	12.66%	0.00%	30.19%	11.93%
0.025	13.44%	145.59%	5%	-4%	20.02%	0.00%	3.87%	2.47%	10.13%	6.64%	15.27%	11.97%	0.37%	19.66%	9.61%
0.025	7.10%	72.41%	5%	-2%	17.41%	1.05%	7.95%	2.75%	11.15%	17.79%	16.65%	8.19%	0.29%	8.90%	4.54%
0.025	0.82%	14.84%	5%	0%	3.60%	10.64%	60.90%	0.78%	0.80%	17.62%	2.40%	1.13%	1.58%	0.39%	0.16%
0.025	10.16%	105.26%	5%	2%	20.11%	0.00%	4.57%	1.27%	14.77%	7.51%	18.92%	11.35%	0.81%	14.92%	5.79%
0.025	17.94%	199.03%	5%	4%	17.06%	0.00%	2.92%	1.17%	10.72%	2.20%	11.77%	12.61%	0.00%	29.90%	11.65%

Table shows optimal results for four different values for s_2: 1, 0.5, 0.1 and 0.025. The value of the first benchmark r_1 is fixed at 5% and the value of the second benchmark, r_2 varies between -4% and 4%. To obtain the optimal upper bound of the solution, the approximating models use scenarios with 1,000 returns by simulating from the mix of empirical and left tail distribution described in (7) with a cutoff point at the 0.1^{st} percentile. Table shows the optimal portfolios average daily returns, standard deviations and the optimal weights on the eleven assets.

Figure 1: Bitcoin Historical Prices

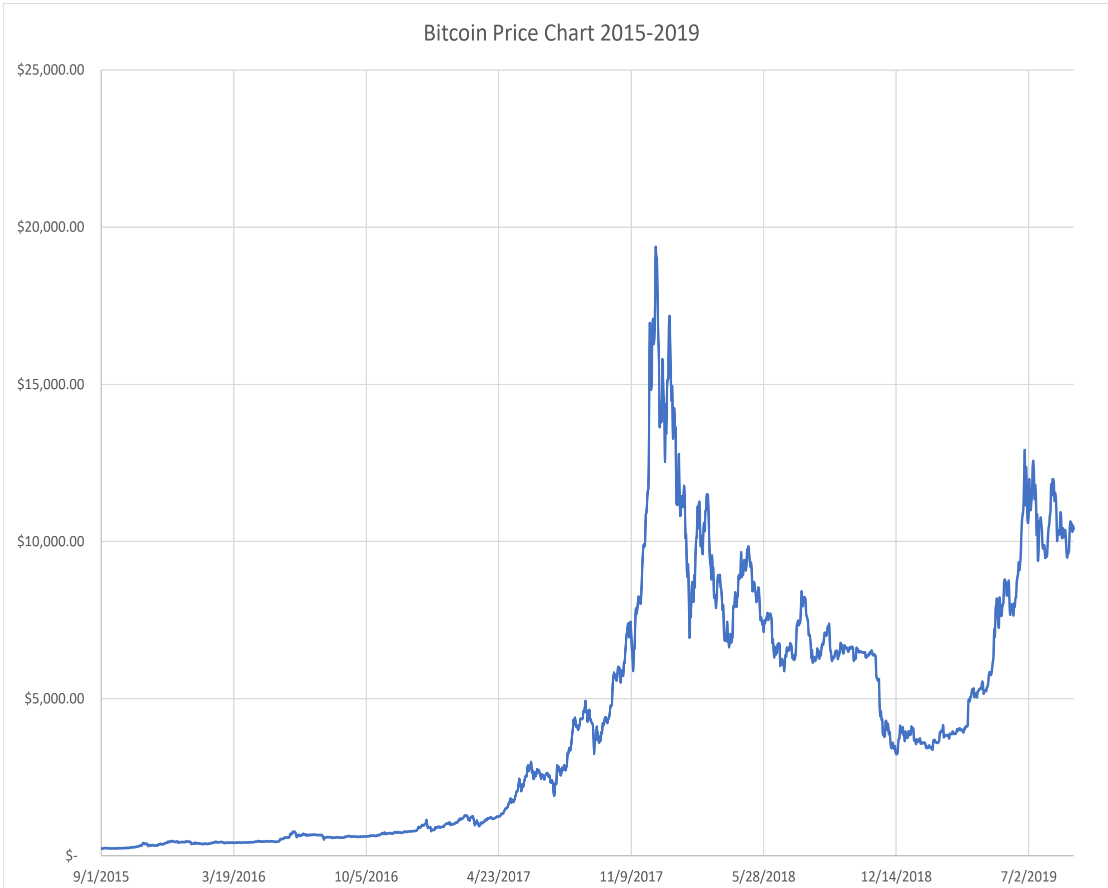

Figure shows the time series of daily Bitcoin prices for the period September 2015-2019. Daily prices are obtained from Yahoo!Finance.

Figure 2: Histogram Bitcoin and S&P 500 (SPY)

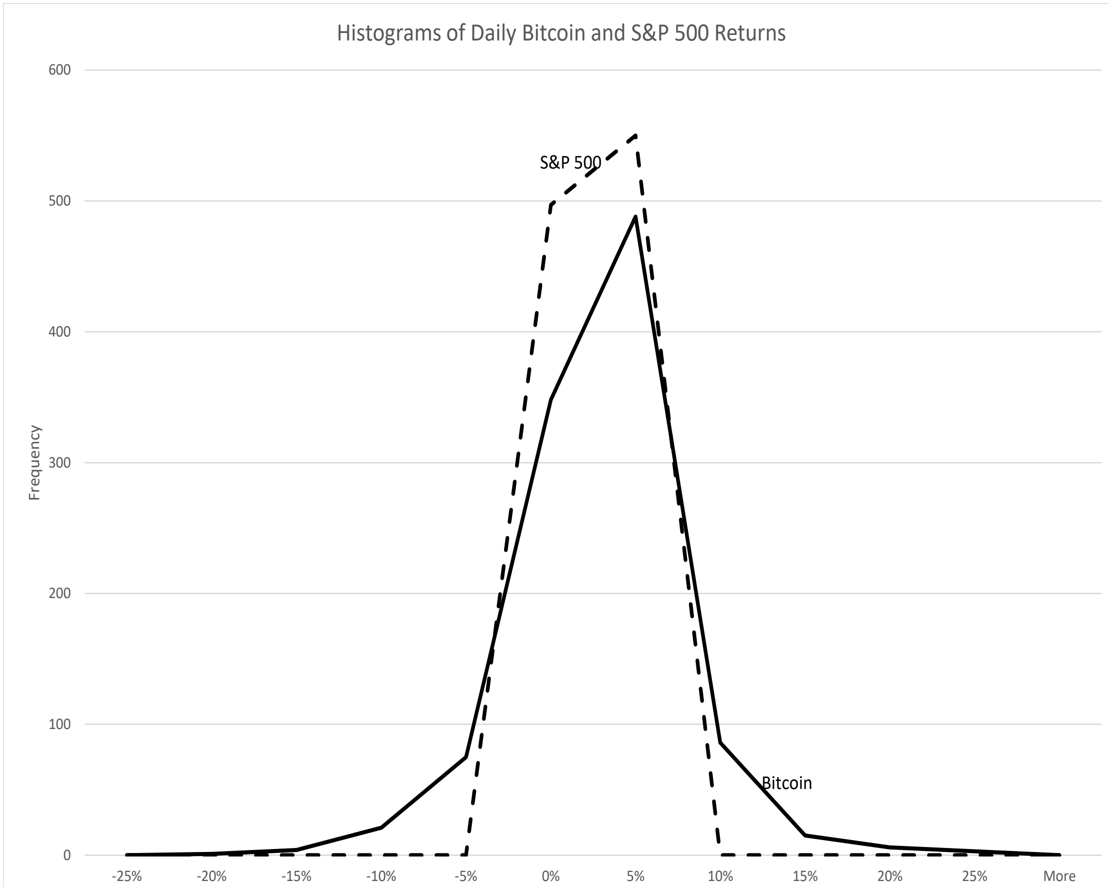

Figure shows histograms for Bitcoin and S&P 500 (SPY) based on daily returns for the period September 2015-2019. SPY daily prices are obtained from Bloomberg and Bitcoin daily prices are obtained from Yahoo!Finance.

Figure 3: Value Function Prospect Theory

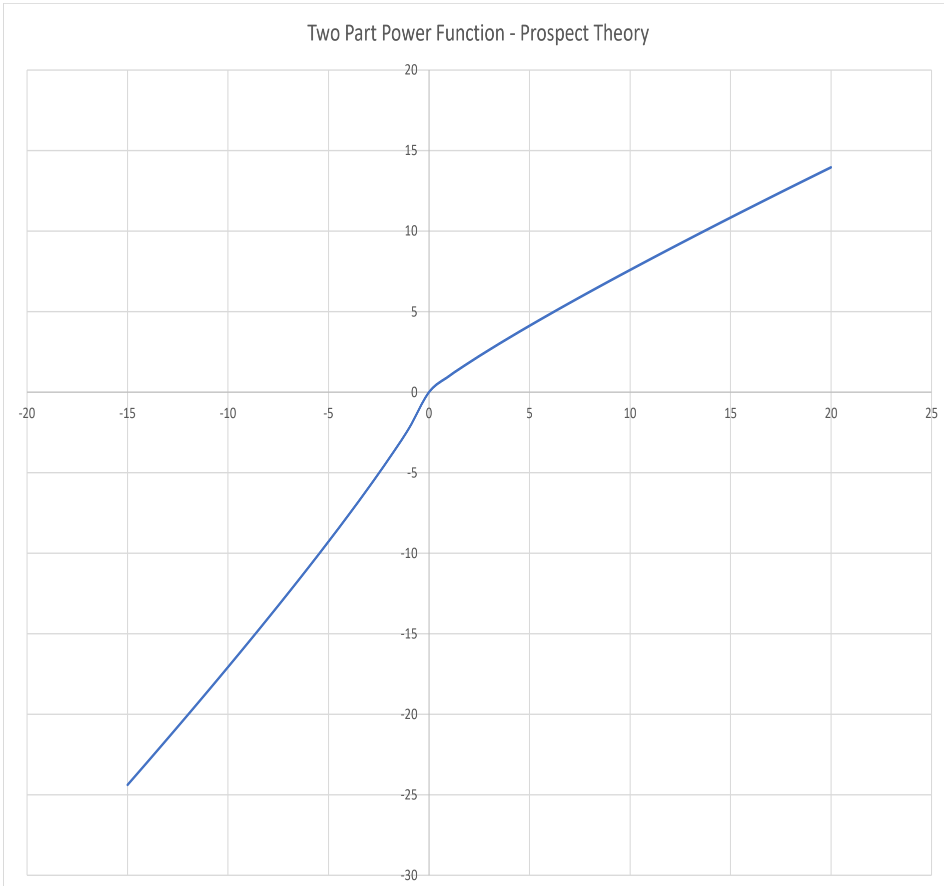

Figure shows a plot of the two part power function from Kahneman and Tversky (1992). The function is: $v(z) = z^{\alpha}$ when $0 < \alpha < 1$ and $z \geq 0$; $v(z) = -\lambda(-z)^{\beta}$ when $\lambda > 1$, $0 < \beta < 1$ and $z < 0$. Kahneman and Tversky (1992) estimated the following values of the parameters: $\alpha = 0.88$, $\beta = 0.88$ and $\lambda = 2.25$.

Figure 4: Linear Piecewise Value Function

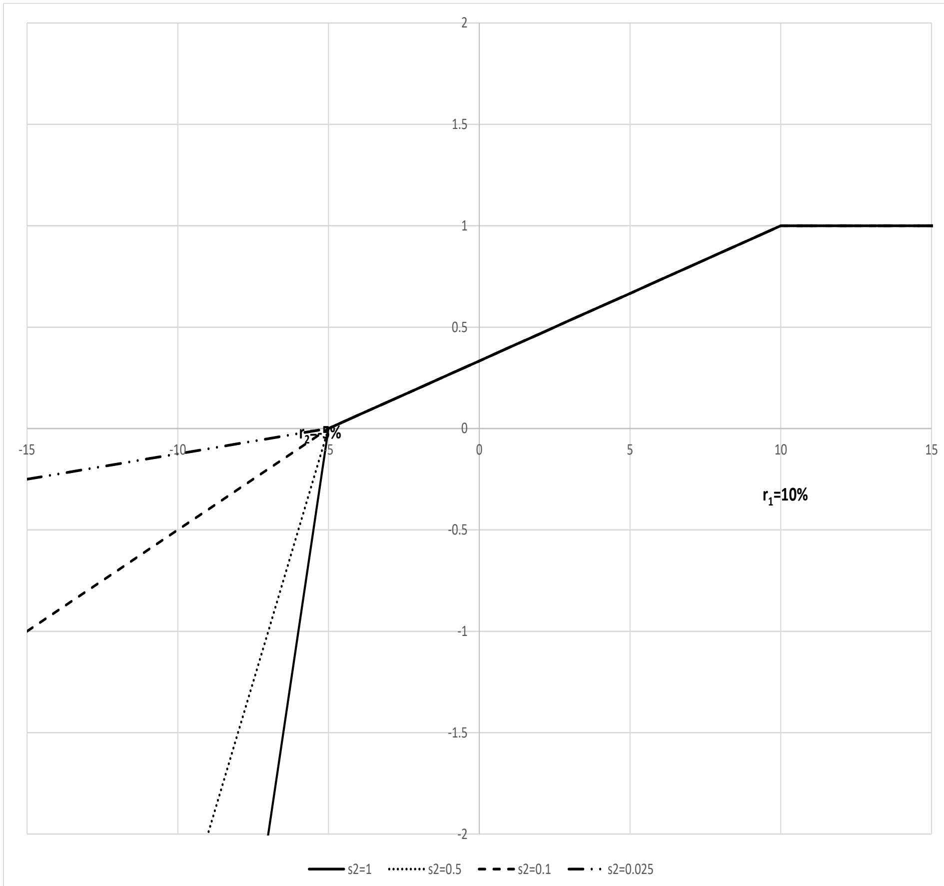

Figure shows plot of the piecewise linear value function where $u(r_1) = 1$, $s_0 = 0$, $u(r_2) = 0$ and $s_1 = 1/(r_1 - r_2)$. The three linear equations describing the value function are: $y = 1$ when $r > r_1$, $y = r/(r_1 - r_2) - r_2/(r_1 - r_2)$ when $r_2 < r < r_1$ and $y = s_2 r - s_2 r_2$ when $r < r_2$. The two benchmarks plotted are: $r_1 = 10\%$ and $r_2 = -5\%$. The values of the slope parameter s_2 are 1, 0.5, 0.1 and 0.025.